Electronic Journal of Qualitative Theory of Differential Equations
Proc. 8th Coll. QTDE, 2008, No. **10** 1-14;
http://www.math.u-szeged.hu/ejqtde/

Two-parametric nonlinear eigenvalue problems

Armands Gritsans, Felix Sadyrbaev *

Abstract

Eigenvalue problems of the form $x'' = -\lambda f(x^+) + \mu g(x^-)$, (i), $x(0) = 0,\ x(1) = 0$, (ii) are considered, where x^+ and x^- are the positive and negative parts of x respectively. We are looking for (λ, μ) such that the problem $(i), (ii)$ has a nontrivial solution. This problem generalizes the famous Fučík problem for piece-wise linear equations. In our considerations functions f and g may be nonlinear functions of super-, sub- and quasi-linear growth in various combinations. The spectra obtained under the normalization condition $|x'(0)| = 1$ are sometimes similar to usual Fučík spectrum for the Dirichlet problem and sometimes they are quite different. This depends on monotonicity properties of the functions $\xi t_1(\xi)$ and $\eta\tau_1(\eta)$, where $t_1(\xi)$ and $\tau_1(\eta)$ are the first zero functions of the Cauchy problems $x'' = -f(x),\ x(0) = 0,\ x'(0) = \xi > 0,\ y'' = g(y)$, $y(0) = 0,\ y'(0) = -\eta,\ (\eta > 0)$ respectively.

1 Introduction

Our goal is to study boundary value problems for two-parameter second order equations of the form

$$x'' = -\lambda f(x^+) + \mu g(x^-), \quad x(0) = 0,\ x(1) = 0, \tag{1}$$

where $f, g : [0, +\infty) \to [0, +\infty)$ are C^1-functions such that $f(0) = g(0) = 0$, $x^+ = \max\{x, 0\},\ x^- = \max\{-x, 0\}$.

The same equation in extended form

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0 \\ \mu g(-x), & \text{if} \quad x < 0. \end{cases} \tag{2}$$

We are motivated by the Fučík equation:

$$x'' = -\lambda x^+ + \mu x^-. \tag{3}$$

*1991 Mathematics Subject Classification. Primary: 34B15.

Key words and phrases: nonlinear spectra, jumping nonlinearity, asymptotically asymmetric nonlinearities, Fučík spectrum.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In extended form:

$$x'' = \begin{cases} -\lambda x, & \text{if} \quad x \ge 0 \\ -\mu x, & \text{if} \quad x < 0, \end{cases} \qquad x(0) = x(1) = 0. \tag{4}$$

The Fučík spectrum is well known and it is depicted in Fig. 1 and Fig. 2 It consists of a set of branches $F_i^{\pm}$, where the number $i = 0, 1, \ldots$ refers to the number of zeros of the respective nontrivial solution in the interval $(0, 1)$ and an upper index, which is either $+$ or $-$, shows either $x'(0)$ is positive or negative.

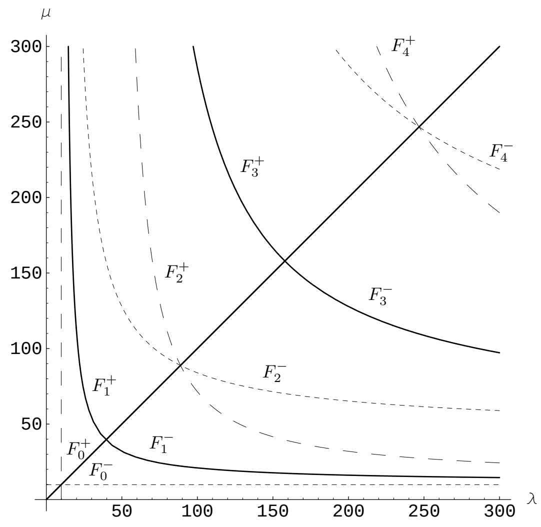

Fig. 1. The classical (λ, μ) Fučík spectrum.

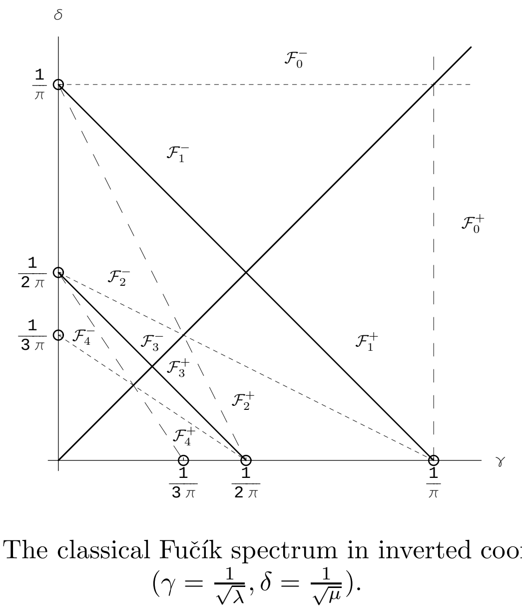

Fig. 2. The classical Fučík spectrum in inverted coordinates $(\gamma = \frac{1}{\sqrt{\lambda}}, \delta = \frac{1}{\sqrt{\mu}})$.

2 One-parametric problems

Consider first the one-parametric eigenvalue problem of the type

$$x'' = -\lambda f(x), \quad x(0) = 0,\ x(1) = 0, \tag{5}$$

where f satisfies our assumptions.

It easily can be seen that this problem may have a continuous spectrum.

For example, the problem

$$x'' = -\lambda x^3, \quad x(0) = 0,\ x(1) = 0.$$

has a positive valued in $(0,1)$ solution $x(t)$ for any $\lambda > 0$. The value $\max_{[0,1]} x(t) := \|x\|$ and λ relate as

$$\|x\| \cdot \lambda = 2\sqrt{2} \cdot \int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

In order to make the problem reasonable one should impose additional conditions. Let us require that

$$|x'(0)| = 1.$$

Let us mention that problems of the type (5) were intensively studied in various settings. For the recent review one may consider the paper [2].

3 Two-parametric problems

3.1 Assumptions

We assume that functions f and g satisfy the following conditions:
(A1) the first zero $t_1(\alpha)$ of a solution to the Cauchy problem

$$u'' = -f(u), \quad u(0) = 0,\ u'(0) = \alpha \tag{6}$$

is finite for any $\alpha > 0$.

Similar property can be assigned to a function g.

We assume that g satisfies the condition:
(A2) the first zero $\tau_1(\beta)$ of a solution to the Cauchy problem

$$v'' = g(-v), \quad v(0) = 0,\ v'(0) = -\beta \tag{7}$$

is finite for any $\beta > 0$.

Functions t_1 and τ_1 are the so called time maps ([5]).

3.2 Formulas for nonlinear Fučík type spectra

Consider

$$x'' = \begin{cases} -\lambda f(x), & \text{if} \quad x \geq 0 \\ \mu g(-x), & \text{if} \quad x < 0, \end{cases} \qquad x(0) = x(1) = 0, \quad |x'(0)| = 1. \tag{8}$$

Let us recall the main result in [4].

Theorem 3.1 *Let the conditions* **(A1)** *and* **(A2)** *hold. The Fučík type spectrum for the problem (8) is given by the relations:*

$$F_0^+ = \Big\{(\lambda, \mu) : \ \lambda \text{ is a solution of } \ \frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1, \quad \mu \geq 0\Big\}, \tag{9}$$

$$F_0^- = \Big\{(\lambda, \mu) : \lambda \geq 0, \ \mu \text{ is a solution of } \ \frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\}, \tag{10}$$

$$F_{2i-1}^+ = \left\{(\lambda; \mu) : \quad i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\right\}, \tag{11}$$

$$F_{2i-1}^- = \left\{(\lambda; \mu) : \quad i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1\right\}, \tag{12}$$

$$F_{2i}^+ = \left\{(\lambda; \mu) : \quad (i+1)\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + i\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\right\}, \tag{13}$$

$$F_{2i}^- = \left\{(\lambda; \mu) : \quad (i+1)\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) + i\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 1\right\}. \tag{14}$$

The same formulas in inverted coordinates $\gamma = \frac{1}{\sqrt{\lambda}}$, $\delta = \frac{1}{\sqrt{\mu}}$ are:

$$\begin{aligned} \mathcal{F}_0^+ &= \big\{(\gamma, \delta) : \ \gamma \text{ is a solution of } \ \gamma\, t_1(\gamma) = 1, \quad \delta > 0\big\} \cup \\ &\quad \cup \big\{(\gamma, \infty) : \ \gamma \text{ is a solution of } \ \gamma\, t_1(\gamma) = 1\big\}, \end{aligned} \tag{15}$$

$$\begin{aligned} \mathcal{F}_0^- &= \big\{(\gamma, \delta) : \gamma > 0, \ \delta \text{ is a solution of } \ \delta\, \tau_1(\delta) = 1\big\} \cup \\ &\quad \cup \big\{(\infty, \delta) : \ \delta \text{ is a solution of } \ \delta\, \tau_1(\delta) = 1\big\}, \end{aligned} \tag{16}$$

$$\mathcal{F}_{2i-1}^+ = \big\{(\gamma; \delta) : \quad i\gamma\, t_1(\gamma) + i\delta\, \tau_1(\delta) = 1, \ \gamma > 0, \ \delta > 0\big\}, \tag{17}$$

$$\mathcal{F}_{2i-1}^- = \big\{(\gamma; \delta) : \quad i\delta\, \tau_1(\delta) + i\gamma\, t_1(\gamma) = 1, \ \gamma > 0, \ \delta > 0\big\}, \tag{18}$$

$$\mathcal{F}_{2i}^+ = \big\{(\gamma; \delta) : \quad (i+1)\gamma\, t_1(\gamma) + i\delta\, \tau_1(\delta) = 1, \ \gamma > 0, \ \delta > 0\big\}, \tag{19}$$

$$\mathcal{F}_{2i}^- = \big\{(\gamma; \delta) : \quad (i+1)\delta\, \tau_1(\delta) + i\gamma\, t_1(\gamma) = 1, \ \gamma > 0, \ \delta > 0\big\}. \tag{20}$$

Corollary 3.1 *The sets* F_{2i-1}^+ *and* F_{2i-1}^- *(respectively* $\mathcal{F}_{2i-1}^+$ *and* $\mathcal{F}_{2i-1}^-$*) coincide.*

Remark 3.1 *Each subset* $F_i^\pm$ *is associated with nontrivial solutions with definite nodal structure. For example, the set*

$$F_4^+ = \left\{(\lambda; \mu) : \quad 3\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + 2\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\right\}$$

is associated with nontrivial solutions that have three positive humps and two negative ones. The total number of interior zeros is exactly four. Similarly, the set

$$F_4^- = \left\{(\lambda;\mu): \quad 2\frac{1}{\sqrt{\lambda}}\,t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + 3\frac{1}{\sqrt{\mu}}\,\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\right\}$$

is associated with nontrivial solutions that have two positive humps and three negative ones.

Remark 3.2 *The additional condition $|x'(0)| = 1$ is not needed if f and g are linear functions (the classical Fučík equation). Then t_1 and τ_1 are constants and do not depend on the initial values of the derivatives.*

3.3 Samples of time maps

Consider equations

$$x'' = -(r+1)x^r, \quad r > 0, \tag{21}$$

which may be integrated explicitly. One has that

$$t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 2A\,\lambda^{\frac{r-1}{2(r+1)}}, \text{ where } A = \int_0^1 \frac{1}{\sqrt{1-\xi^{r+1}}}\,d\xi, \tag{22}$$

so t_1 is decreasing in λ for $r \in (0,1)$,
  t_1 is constant for $r = 1$,
  t_1 is increasing in λ for $r > 1$.

The function

$$u(\lambda) = \frac{1}{\sqrt{\lambda}}t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 2A\,\lambda^{-\frac{1}{r+1}}$$

is decreasing for $r > 0$.

4 Some properties of spectra

Introduce the functions

$$u(\lambda) := \frac{1}{\sqrt{\lambda}}t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) \quad v(\mu) := \frac{1}{\sqrt{\mu}}\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big), \tag{23}$$

where t_1 and τ_1 are the time maps associated with f and g respectively. Due to Theorem 3.1 the spectrum of the problem (8) is a union of pairs (λ, μ) such that one of the relations

$$\begin{array}{llllll} u(\lambda) & + & v(\mu) & = 1, & F_1^{\pm} \\ 2u(\lambda) & + & v(\mu) & = 1, & F_2^{+} \\ u(\lambda) & + & 2v(\mu) & = 1, & F_2^{-} \\ 2u(\lambda) & + & 2v(\mu) & = 1, & F_3^{\pm} \\ 3u(\lambda) & + & 2v(\mu) & = 1, & F_4^{+} \\ 2u(\lambda) & + & 3v(\mu) & = 1, & F_4^{-} \\ \dots \end{array} \tag{24}$$

holds. The coefficients at $u(\lambda)$ and $v(\mu)$ indicate the numbers of "positive" and "negative" humps of the respective eigenfunctions.

4.1 Monotone functions u and v

Suppose that both functions u and v are monotonically decreasing. Then the same do the multiples iu and iv, i is a positive integer.

Theorem 4.1 *Suppose that the functions u and v monotonically decrease from $+\infty$ to zero. Then the spectrum of the problem (8) is essentially the classical Fučík spectrum, that is, it is a union of branches $F_i^{\pm}$, which are the straight lines for $i = 0$, the curves which look like hyperbolas and have both vertical and horizontal asymptotes, for $i > 0$.*

Proof. First of all notice that the value $u(\lambda) = \frac{1}{\sqrt{\lambda}} t_1\left(\frac{1}{\sqrt{\lambda}}\right)$ is exactly the distance between two consecutive zeros of a solution to the problem $x'' = -\lambda f(x)$, $x(0) = 0$, $x'(0) = 1$. Similarly the value $v(\mu) = \frac{1}{\sqrt{\mu}} \tau_1\left(\frac{1}{\sqrt{\mu}}\right)$ is the distance between two consecutive zeros of a solution to the problem $y'' = \mu g(-y)$, $y(0) = 0$, $y'(0) = -1$.

Let λ_1, λ_2, λ_3 and so on be the points of intersection of $u(\lambda), 2u(\lambda), 3u(\lambda), \ldots$ with the horizontal line $u = 1$. Respectively μ_1, μ_2, μ_3 and so on for the function $v(\mu)$ (see the Fig. 3 and Fig. 4).

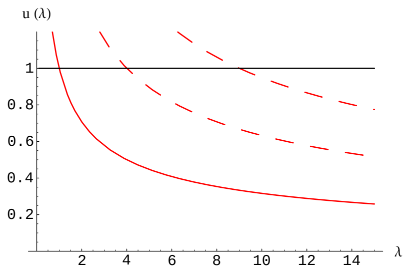

Fig. 3. The graphs of $u(\lambda), 2u(\lambda), 3u(\lambda)$ (schematically).

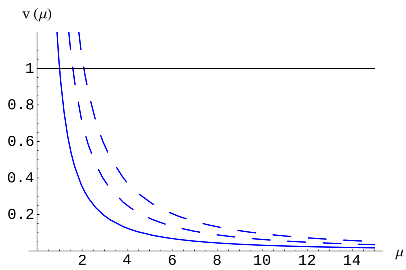

Fig. 4. The graphs of $v(\mu), 2v(\mu), 3v(\mu)$ (schematically).

Positive solutions to the problem with no zeros in the interval $(0, 1)$ appear for $\lambda = \lambda_1$. Thus F_0^+ is a straight line $\{(\lambda_1, \mu) : \mu \geq 0\}$. Similarly F_0^- is a straight line $\{(\lambda, \mu_1) : \lambda \geq 0\}$.

The branches $F_1^{\pm}$ which are defined by the first equation of (24) coincide and look like hyperbola with the vertical asymptote at $\lambda = \lambda_1$ and horizontal asymptote at $\mu = \mu_1$.

The branch F_2^+ has the vertical asymptote at $\lambda = \lambda_2$ and horizontal asymptote at $\mu = \mu_1$. This can be seen from the second equation of (24).

The branch F_2^- has the vertical asymptote at $\lambda = \lambda_1$ and horizontal asymptote at $\mu = \mu_2$. This is a consequence of the third equation of (24). Notice that the branches F_2^+ and F_2^- need not to cross at the bisectrix $\lambda = \mu$ unless $g \equiv f$ (in contrast with the case of the classical Fučík spectrum).

The branches $F_3^\pm$ coincide and have the vertical asymptote at $\lambda = \lambda_2$ and horizontal asymptote at $\mu = \mu_2$.

The branch F_4^+ has the vertical asymptote at $\lambda = \lambda_3$ and horizontal asymptote at $\mu = \mu_2$.

The branch F_4^- has the vertical asymptote at $\lambda = \lambda_2$ and horizontal asymptote at $\mu = \mu_3$. The branches F_4^+ and F_4^- need not to cross at the bisectrix.

In a similar manner any of the remaining branches can be considered. $\square$

Proposition 4.1 *The function* $u(\lambda) = \frac{1}{\sqrt{\lambda}} t_1\left(\frac{1}{\sqrt{\lambda}}\right)$, *where* t_1 *is defined in (6), is monotonically decreasing if*

$$1 - \frac{F(x)F''(x)}{f^2(x)} > 0, \quad F(x) = \int_0^x f(s)\,ds. \tag{25}$$

Proof. Let us show that the function $\alpha\, t_1(\alpha)$ is monotonically increasing for $\alpha > 0$. Consider the Cauchy problem $x'' + f(x) = 0$, $x(0) = 0$, $x'(0) = \alpha$. A solution x satisfies the relation $\frac{1}{2}x'^2(t) + F(x(t)) = h$, where $h = \frac{1}{2}\alpha^2 = F(x_+)$, x_+ is a maximal value of $x(t)$.

It was shown in [3, Lemma 2.1] that the function $T(h) = 2\int_0^{x_+} \frac{ds}{\sqrt{2(h-F(s))}}$ has the derivative

$$\frac{dT}{dh} = \frac{2}{h}\int_0^{x_+} \Big(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\Big)\frac{dx}{\sqrt{2(h - F(x))}}. \tag{26}$$

Notice that $t_1(\alpha) = T(\frac{1}{2}\alpha^2)$. One has that

$$\begin{aligned}
[\alpha\, t_1(\alpha)]'_\alpha &= t_1(\alpha) + \alpha\, t_1'(\alpha) \\
&= 2\int_0^{x_+} \frac{dx}{\sqrt{\alpha^2 - 2F(x))}} + 4\int_0^{x_+}\left(\frac{1}{2} - \frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{\alpha^2 - 2F(x))}} \\
&= 4\int_0^{x_+}\left(1 - \frac{F(x)F''(x)}{f^2(x)}\right)\frac{dx}{\sqrt{\alpha^2 - 2F(x))}}
\end{aligned} \tag{27}$$

$\square$

For instance, if $x'' + x = 0$, then $f = x$, $F = \frac{1}{2}x^2$, $\omega(\alpha) := \alpha t_1(\alpha) = \pi\alpha$, $\omega' = \pi$. Taking into account that $x_+ = \alpha$ one obtains from (27)

$$\omega'(\alpha) = 4\int_0^\alpha (1 - \frac{1}{2})\frac{dx}{\sqrt{\alpha^2 - x^2}} = 2\arcsin\frac{x}{\alpha}\Big|_0^\alpha = \pi.$$

4.2 Non-monotone functions u and v

It is possible that the functions $u(\lambda) = \frac{1}{\sqrt{\lambda}} t_1\left(\frac{1}{\sqrt{\lambda}}\right)$ and $v(\mu) = \frac{1}{\sqrt{\mu}} \tau_1\left(\frac{1}{\sqrt{\mu}}\right)$ are not monotone.

Then spectra may differ essentially from those in the monotone case.

Proposition 4.2 *Suppose that $u(\lambda)$ and $v(\mu)$ are not zeros at $\lambda = 0$ and $\mu = 0$ respectively and monotonically decrease to zero starting from some values $\lambda_\bigstar$ and $\mu_\bigstar$. Then the subsets $F_i^{\pm}$ of the spectrum behave like the respective branches of the classical Fučík spectrum for large numbers i, that is, they form curves looking like hyperbolas which have vertical and horizontal asymptotes.*

Indeed, notice that for large enough values of i the functions $iu(\lambda)$ and $iv(\mu)$ monotonically decrease to zero in the regions $\{\lambda \geq \lambda_\Delta, 0 < u < 1\}$, $\{\mu \geq \mu_\Delta, 0 < v < 1\}$ respectively (for some λ_Δ and μ_Δ) and are greater than unity for $0 < \lambda < \lambda_\Delta$ and $0 < \mu < \mu_\Delta$ respectively. Therefore one may complete the proof by analyzing the respective relations in (24).

If one (or both) of the functions u and v is non-monotone then the spectrum may differ essentially from the classical Fučík spectrum. Consider the case depicted in Fig. 5.

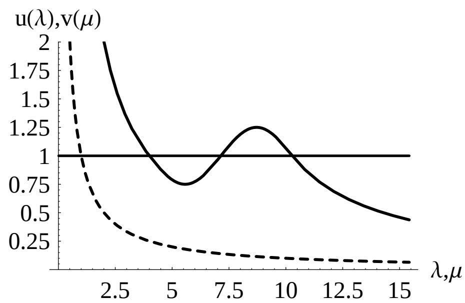

Fig. 5. Functions u (solid line) and v (dashed line).

Proposition 4.3 *Let the functions u and v behave like depicted in Fig. 5, that is, v monotonically decreases from $+\infty$ to zero and u has three segments of monotonicity, u tends to zero as λ goes to $+\infty$. Then the subset $F_1^{\pm}$ consists of two components.*

Indeed, let λ_1, λ_2 and λ_3 be successive points of intersection of the graph of u with the line $u = 1$. Denote λ_* the point of minimum of $u(\lambda)$ in the interval (λ_1, λ_2). Let μ_* be such that $u(\lambda_*) + v(\mu_*) = 1$. It is clear that there exists a U-shaped curve with vertical asymptotes at $\lambda = \lambda_1$ and $\lambda = \lambda_2$ and with a minimal value μ_* at λ_* which belongs to F_1^+. There also exists a hyperbola looking curve with the vertical asymptote at $\lambda = \lambda_3$ and horizontal asymptote at $\mu = \mu_1$, where μ_1 is the (unique) point of intersection of the graph of v with the line $v = 1$.

There are no more points belonging to F_1^+.

5 Examples

Let
$$0 < a_1 < a_2 < a_3, \quad b_1 > b_2 > 0, \quad b_3 > b_2.$$

Consider a piece-wise linear function:
$$f(x) = \begin{cases} f_1(x), & 0 \le x \le a_1, \\ f_2(x), & a_1 \le x \le a_2, \\ f_3(x), & x \ge a_3, \end{cases} \tag{28}$$
$$f_1(x) = p_1 x + q_1, \quad f_2(x) = p_2 x + q_2, \quad f_3(x) = p_3 x + q_3,$$
$$f_1(0) = 0, \quad f_1(a_1) = f_2(a_1), \quad f_2(a_2) = f_3(a_2), \quad f_3(a_3) = b_3.$$

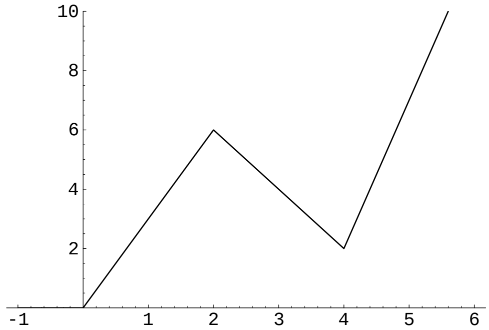

Fig. 6. Function $f(x)$.

Notice that
$$p_1 = \frac{b_1}{a_1}, \qquad q_1 = 0,$$
$$p_2 = \frac{b_2 - b_1}{a_2 - a_1}, \qquad q_2 = \frac{b_1 a_2 - a_1 b_2}{a_2 - a_1},$$
$$p_3 = \frac{b_3 - b_2}{a_3 - a_2}, \qquad q_3 = \frac{b_2 a_3 - a_2 b_3}{a_3 - a_2}.$$

Let $t_1(\alpha)$ be the first positive zero of a solution to the initial value problem
$$x'' = -f(x), \quad x(0) = 0, \quad x'(0) = \alpha > 0. \tag{29}$$

Denote $F(x) = \int_0^x f(s)\,ds$. Direct calculations ([1]) show that

1. if $0 \le \alpha \le \sqrt{2F(a_1)}$, then $t_1(\alpha) = \pi\sqrt{\frac{a_1}{b_1}}$;

2. if $\sqrt{2F(a_1)} \le \alpha \le \sqrt{2F(a_2)}$, then
$$t_1(\alpha) = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha} + \\ + \sqrt{\frac{a_2 - a_1}{b_1 - b_2}} \ln \frac{D_2(\alpha)}{\left(-2b_1 + 2\sqrt{\frac{b_1 - b_2}{a_2 - a_1}}\sqrt{\alpha^2 - a_1 b_1}\right)^2},$$

3. if $\alpha \geq \sqrt{2F_2(a_2)}$, then

$$t_1(\alpha) = 2\sqrt{\frac{a_1}{b_1}} \arcsin \frac{\sqrt{a_1 b_1}}{\alpha} + \sqrt{\frac{a_3 - a_2}{b_3 - b_2}} \left[\pi - 2 \arcsin \frac{2b_2}{\sqrt{D_3(\alpha)}} \right] +$$

$$+ 2\sqrt{\frac{a_2 - a_1}{b_1 - b_2}} \ln \left| \frac{-b_2 + \sqrt{\frac{b_1 - b_2}{a_2 - a_1}} \sqrt{\alpha^2 - a_1 b_1 - (a_2 - a_1)(b_1 + b_2)}}{-b_1 + \sqrt{\frac{b_1 - b_2}{a_2 - a_1}} \sqrt{\alpha^2 - a_1 b_1}} \right|,$$

where

$$D_2(\alpha) = 4\,\frac{b_1 - b_2}{a_1 - a_2}\alpha^2 + 4b_1\,\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}, \quad D_3(\alpha) = 4\,\frac{b_2 - b_3}{a_2 - a_3}\alpha^2 +$$

$$+ 4\,\frac{-a_2 b_1 b_2 + a_1 b_2^2 + a_3 b_2^2 + a_2 b_1 b_3 - a_1 b_2 b_3 + a_2 b_2 b_3}{a_2 - a_3}.$$

The first zero function is asymptotically linear:

$$\lim_{\alpha \to +\infty} t_1(\alpha) = \sqrt{\frac{a_3 - a_2}{b_3 - b_2}}\pi.$$

Consider equation

$$x'' = -\lambda f(x^+) + \mu f(x^-),$$

where $f(x)$ is a piece-wise linear function depicted in Fig. 6. Let parameters of the piece-wise linear function $f(x)$ be

$$a_1 = 0.1, \qquad a_2 = 0.3, \qquad a_3 = 0.31,$$
$$b_1 = 9, \qquad b_2 = 0.5, \qquad b_3 = 150.$$

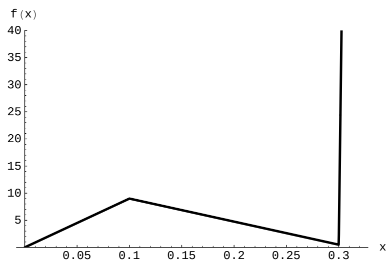

Fig. 7. The graph of $y = f(x)$.

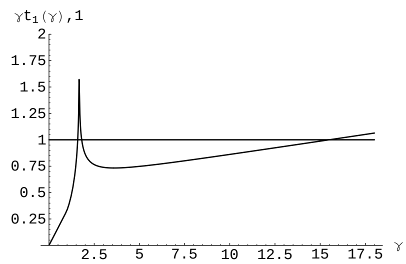

Fig. 8. The graphs of $y = \gamma t_1(\gamma)$ $(\gamma = \frac{1}{\sqrt{\lambda}})$ and $y = 1$.

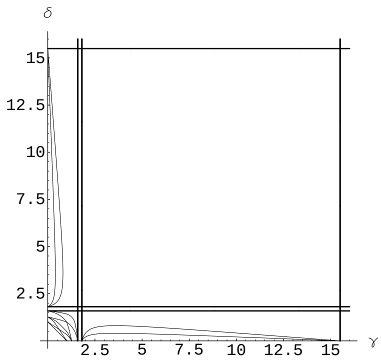

Fig. 9. The subset $\mathcal{F}_0^+$ in the (γ, δ)-plane.

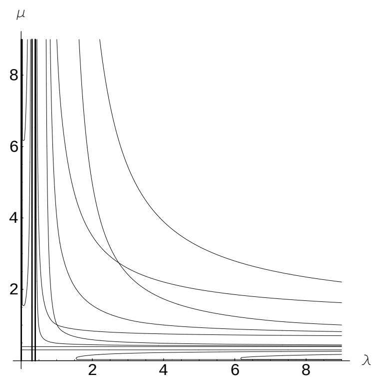

Fig. 10. The subset F_0^+ in the (λ, μ)-plane.

The subset F_0^+ consists of three vertical lines which correspond to three solutions of the equation $\frac{1}{\sqrt{\lambda}} t_1(\frac{1}{\sqrt{\lambda}}) = 1$.

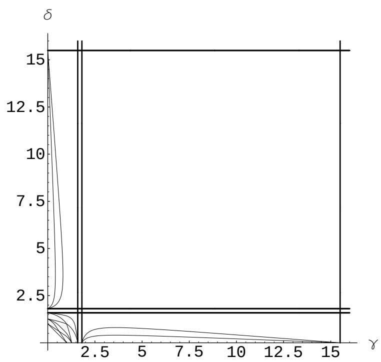

Fig. 11. The subset $\mathcal{F}_0^-$ in the (γ, δ)-plane.

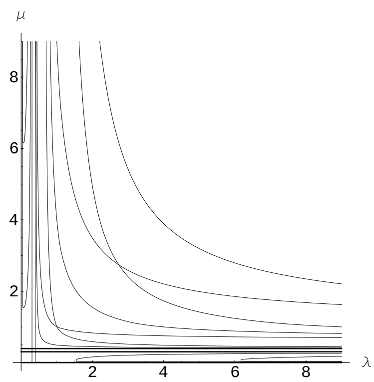

Fig. 12. The subset F_0^- in the (λ, μ)-plane.

The subset F_0^- consists of horizontal lines which correspond to solutions of the equation $\frac{1}{\sqrt{\mu}} \tau_1(\frac{1}{\sqrt{\mu}}) = 1$.

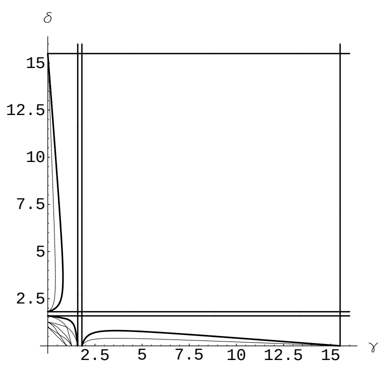

Fig. 13. The subset $\mathcal{F}_1^+ = \mathcal{F}_1^-$ in the (γ, δ)-plane.

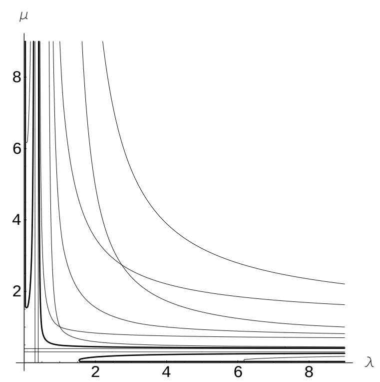

Fig. 14. The subset $F_1^+ = F_1^-$ in the (λ, μ)-plane.

Properties of the subsets $F_1^\pm$ depend on solutions of the equation

$$u(\lambda) + v(\mu) = 1.$$

A set of solutions of this equation consists of exactly three components due to non-monotonicity of the functions $u(\lambda)$ and $v(\mu)$. Respectively, properties of the subsets $\mathcal{F}_1^\pm$ depend on solutions of the equation

$$\gamma t_1(\gamma) + \delta \tau_1(\delta) = 1.$$

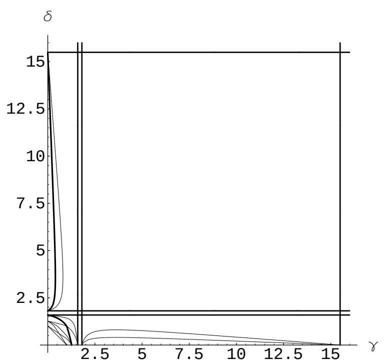

Fig. 15. The subset $\mathcal{F}_2^+$ in the (γ, δ)-plane.

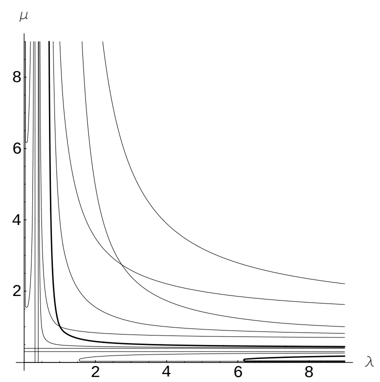

Fig. 16. The subset F_2^+ in the (λ, μ)-plane.

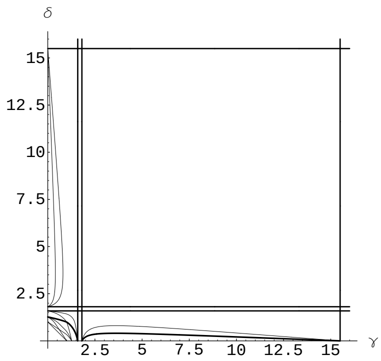

Fig. 17. The subset $\mathcal{F}_2^-$ in the (γ, δ)-plane.

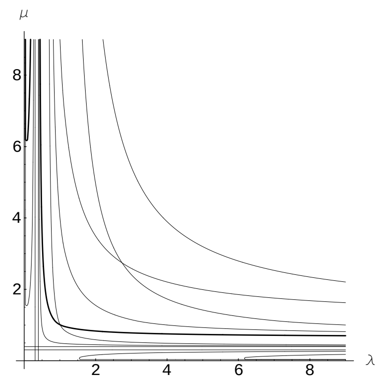

Fig. 18. The subset F_2^- in the (λ, μ)-plane.

The subsets $F_2^{\pm}$ look a little bit different since now their properties depend on a set of solutions of equations

$$2\frac{1}{\sqrt{\lambda}}t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + \frac{1}{\sqrt{\mu}}\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1$$

and

$$\frac{1}{\sqrt{\lambda}}t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + 2\frac{1}{\sqrt{\mu}}\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1.$$

Respectively, properties of the subsets $\mathcal{F}_2^{\pm}$ depend on solutions of equations

$$2\gamma t_1(\gamma) + \delta\tau_1(\delta) = 1$$

and

$$\gamma t_1(\gamma) + 2\delta\tau_1(\delta) = 1.$$

References

[1] A. Gritsans and F. Sadyrbaev. On nonlinear eigenvalue problems. *Proceedings Inst. Math. Comp. Sci. of the University of Latvia "Mathematics. Differential Equations", vol. 6 (2006), 76 - 86.* http://www.lumii.lv/sbornik2006/Sbornik-2006-english.htm

[2] P. Korman. Global solution branches and exact multiplicity of solutions for two-point boundary value problems. *In: Handbook of Diff. Equations, ODE, Vol. III. Elsevier - North Holland, Amsterdam, 2006, 548 - 606.*

[3] Bin Liu. On Littlewood's boundedness problem for sublinear Duffing equations. *Trans. Amer. Math. Soc., 2000, 353, No. 4, 1567 - 1585.*

[4] F. Sadyrbaev and A. Gritsans. Nonlinear Spectra for Parameter Dependent Ordinary Differential Equations. *Nonlinear Analysis: Modelling and Control, 2007, 12, No.2, 253 - 267.*

[5] R. Schaaf. Global solution branches of two-point boundary value problems. *Lect. Notes Math. 1458. Springer-Verlag, Berlin - Heidelberg - New York, 1990.*

(Received August 31, 2007)

Faculty of Natural Sciences and Mathematics (DMF), Daugavpils University,
Parades str. 1, 5400 Daugavpils, Latvia
E-mail addresses: felix@latnet.lv (the address for communication), arminge@inbox.lv